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Weighted Means and Summability by the Circle and Other Methods*

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It is proved that if the weighted means of a sequence satisfy certain order conditions, then the sequence is summable by every non-trivial circle method, by the Cesàro method, C_1, and by the Borel-type method (B, α, β).

1. INTRODUCTION

Suppose throughout that $\{s_n\}$ is a given sequence, and that $\{q_n\}$ is a sequence of positive numbers. The sequence of weighted means $\{t_n\}$ is defined by

$$t_n := \frac{1}{Q_n} \sum_{k=0}^{n} q_k s_k, \qquad \text{where} \quad Q_n := \sum_{k=0}^{n} q_k.$$

The sequence $\{s_n\}$ is said to be summable to s by the weighted mean method M_q if $t_n \to s$. In particular, the Cesàro method C_1 and the logarithmic method l are the methods M_q with $q_k := 1$ and $q_k := 1/(k+1)$, respectively. The sequence of C_1-means of any sequence $\{x_n\}$ will be denoted by $\{x_n^1\}$.

Recall that the Borel-type method (B, α, β), the Valiron method V_α $(\alpha > 0)$, the Euler method E_δ, the Meyer–König method S_δ, and the Taylor method T_δ $(0 < \delta < 1)$ are defined by

$$s_n \to s(B, \alpha, \beta) \qquad \text{if} \quad \alpha e^{-x} \sum_{n=N}^{\infty} s_n \frac{x^{\alpha n + \beta - 1}}{\Gamma(\alpha + \beta)} \to s \quad \text{as} \quad x \to \infty,$$

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where $\alpha N+\beta>0$;

$$s_n \to s(V_\alpha) \quad \text{if} \quad \left(\frac{\alpha}{2\pi n}\right)^{1/2} \sum_{k=0}^{\infty} \exp\left(-\frac{\alpha(n-k)^2}{2n}\right) s_k \to s \quad \text{as} \quad n\to\infty;$$

$$s_n \to s(E_\delta) \quad \text{if} \quad \sum_{k=0}^{n} \binom{n}{k} \delta^k (1-\delta)^{n-k} s_k \to s \quad \text{as} \quad n\to\infty;$$

$$s_n \to s(S_\delta) \quad \text{if} \quad (1-\delta)^{n+1} \sum_{k=0}^{\infty} \binom{n+k}{k} \delta^k s_k \to s \quad \text{as} \quad n\to\infty;$$

$$s_n \to s(T_\delta) \quad \text{if} \quad (1-\delta)^{n+1} \sum_{k=0}^{\infty} \binom{n+k}{k} \delta^k s_{n+k} \to s \quad \text{as} \quad n\to\infty.$$

Let Γ denote the family of all the methods (B, α, β) and V_α with $\alpha>0$, and all the methods E_δ, S_δ, and T_δ with $0<\delta<1$. The non-Borel-type methods in Γ are commonly called "circle" methods. All the methods in Γ are regular and not equivalent to convergence. For basic properties of these methods see [1, 4, 6, 9].

In general C_1-summability does not imply summability by any member of Γ. However, the following result is known [7, Theorem 4; 2, Theorems 1, 2, and 3].

THEOREM C. *If $s_n^1 = s + o(n^{-1/2})$, then $\{s_n\}$ is summable to s by every member of Γ.*

It is also known [4, p. 59] that C_1-summability implies l-summability, but that l-summability does not imply C_1-summability. Parameswaran has proved [8, Theorem 1] the following theorem.

THEOREM L. *If*

$$\frac{1}{\log n} \sum_{k=0}^{n} \frac{s_k}{k+1} = s + \frac{\mu}{\log n} + o\left(\frac{n^{-1/2}}{\log n}\right),$$

then $\{s_n\}$ is summable to s by every member of Γ (and by C_1).

It is easily seen that $\log n$ can be replaced by $Q_n := \sum_{k=0}^{n} 1/(k+1)$ in the above hypothesis. Thus Theorem L shows that l-summability together with an order relation implies C_1-summability.

2. Results

It is the aim of this paper to establish the following two theorems.

THEOREM 1. *Suppose that*

$$\frac{q_{n-1}}{q_n} = 1 + O(n^{-1}) \tag{1}$$

and

$$t_n = s + \frac{\mu}{Q_n} + o\left(\frac{n^{1/2} q_n}{Q_n}\right), \tag{2}$$

where s and μ are constants. Then $\{s_n\}$ is summable to s by every member of Γ (and by C_1). Moreover, o cannot be replaced by O in (2). Indeed, there exists a C_1-summable bounded sequence $\{s_n\}$ which is not summable by any member of Γ and which satisfies

$$t_n = O\left(\frac{n^{1/2} q_n}{Q_n}\right).$$

This shows that M_q-summability together with an order condition implies summability by every member of Γ and that the order condition is best possible in a strong sense. If the sequence $\{q_n\}$ satisfies $n^{1/2} q_n \to \infty$, then the term μ/Q_n in (2) can be absorbed into the term $o(n^{1/2} q_n/Q_n)$, and so Theorem C is special case of Theorem 1.

THEOREM 2. *Suppose that condition (1) holds, and that*

$$n q_n s_n = O_L(1), \tag{3}$$

$$\frac{1}{n q_n} = O(1), \tag{4}$$

and

$$\frac{1}{n} \sum_{k=1}^{n} k^{1/2} t_k Q_k = \mu + o(n^{-1/2}), \tag{5}$$

where μ is a constant. Then $\{s_n\}$ is sumable to 0 by every member of Γ.

The notation $x_n = O_L(1)$ signifies, as usual, that $\liminf x_n > -\infty$. Theorem 2 generalizes another of Parameswaran's results [8, Theorem 2]. His result is essentially the case $q_n := 1/n$ of our Theorem 2. The first conclusion of Theorem 1 is an immediate consequence of Theorem C and the following lemma.

LEMMA. *Suppose that conditions* (1) *and* (2) *hold. Then*

$$s_n^1 = s + o(n^{-1/2}).$$

Proof. We may suppose that $s = 0$. Let

$$\tau_n := t_n Q_n.$$

Then, for $n \geqslant 1$,

$$s_n = \frac{\tau_n - \mu}{q_n} - \frac{\tau_{n-1} - \mu}{q_{n-1}} - (\tau_{n-1} - \mu)\left(\frac{1}{q_n} - \frac{1}{q_{n-1}}\right),$$

so that

$$\sum_{k=0}^{n} s_k = \frac{\tau_n - \mu}{q_n} - \sum_{k=1}^{n} (\tau_{k-1} - \mu)\left(\frac{1}{q_k} - \frac{1}{q_{k-1}}\right) + \frac{\mu}{q_0} \tag{6}$$

and so

$$s_n^1 = o(n^{-1/2}) - \frac{1}{n+1} \sum_{k=1}^{n} (\tau_{k-1} - \mu)\left(\frac{1}{q_k} - \frac{1}{q_{k-1}}\right). \tag{7}$$

Next, by (2), we have that $\tau_k = \mu + \varepsilon_k (k+1)^{1/2} q_k$, where $\varepsilon_k \to 0$. Hence, by (1), we get that

$$\begin{aligned} n^{-1/2} &\sum_{k=1}^{n} (\tau_{k-1} - \mu)\left(\frac{1}{q_k} - \frac{1}{q_{k-1}}\right) \\ &= n^{-1/2} \sum_{k=1}^{n} k^{-1/2} \varepsilon_{k-1} k \left(\frac{q_{k-1}}{q_k} - 1\right) = o(1). \end{aligned} \tag{8}$$

It follows from (7) and (8) that $s_n^1 = o(n^{-1/2})$. ∎

Proof of Theorem 1. As stated above, the first conclusion follows from the lemma and Theorem C. To prove the remaining conclusions we observe, as Parameswaran did [8, proof of Theorem 1], that a consequence of a result due to Lorentz [5] is that there exists a bounded sequence $\{x_n\}$ satisfying $x_n^1 = O(n^{-1/2})$ which is not E_δ-summable for any $\delta \in (0, 1)$. We now define $\{s_n\}$ so that $\tau_n := t_n Q_n = (n+1) q_n x_n^1$. Then

$$\tau_n = O(n^{1/2} q_n). \tag{9}$$

It follows from (6) with $\mu=0$ that

$$s_n^1 = x_n^1 - y_n^1, \qquad \text{where} \quad y_n := \tau_{n-1}\left(\frac{1}{q_n} - \frac{1}{q_{n-1}}\right)$$

$$\text{for} \quad n \geqslant 1 \quad \text{and} \quad y_0 := 0,$$

and so

$$s_n = x_n - y_n.$$

Next, by (9) and (1), we have that

$$y_n = \frac{\tau_{n-1}}{q_{n-1}}\left(\frac{q_{n-1}}{q_n} - 1\right) = O(n^{-1/2}) = o(1)$$

so that $\{y_n\}$ is E_δ-summable. Since $\{x_n\}$ is bounded and not E_δ-summable, it follows that $\{s_n\}$ also has these properties. Now it is known [6, Satz 25; 3, Theorem 3] that all members of Γ are equivalent for bounded sequences, and so $\{s_n\}$ cannot be summable by any member of Γ. ▌

Proof of Theorem 2. For $n \geqslant 1$, let

$$z_n := \tau_n - \mu n^{-1/2}, \qquad \text{where} \quad \tau_n := t_n Q_n.$$

Then we are given that

$$\frac{1}{n}\sum_{k=1}^{n} k^{1/2} z_k = o(n^{-1/2}). \tag{10}$$

By [2, Theorem 1 with $p=1$] and (10),

$$n^{1/2} z_n \to 0(B, \alpha, \beta). \tag{11}$$

Further, (10) implies that

$$z_n = o(1). \tag{12}$$

Next, using the notation $\Delta x_n := x_n - x_{n-1}$, we have that

$$\begin{aligned}
\Delta(n^{1/2} z_n) &= n^{1/2}(z_n - z_{n-1}) + z_{n-1}(n^{1/2} - (n-1)^{1/2}) \\
&= n^{1/2}[(\tau_n - \mu n^{-1/2}) - (\tau_{n-1} - \mu(n-1)^{-1/2})] \\
&\quad + z_{n-1}(n^{1/2} - (n-1)^{1/2}) \\
&= n^{1/2}(\tau_n - \tau_{n-1}) + \mu n^{1/2}((n-1)^{-1/2} - n^{-1/2}) \\
&\quad + z_{n-1}(n^{1/2} - (n-1)^{1/2}).
\end{aligned}$$

Hence, by (12) and (3),

$$n^{1/2}\Delta(n^{1/2}z_n) = nq_n s_n + o(1) = O_L(1). \tag{13}$$

By virtue of a Tauberian theorem for Borel-type methods [3, Theorem 1 with $r=0$], it follows from (11) and (13) that

$$n^{1/2}z_n = o(1). \tag{14}$$

At this stage it is worth noting that if $\mu = 0$, then (14) and (4) imply that $\tau_n = o(n^{1/2}q_n)$, so that the required conclusion follows from Theorem 1. Returning to the general case, we deduce from (6) that

$$\begin{aligned} s_n^1 &= \frac{\tau_n}{(n+1)q_n} - \frac{1}{n+1}\sum_{k=1}^{n} \tau_{k-1}\left(\frac{1}{q_k} - \frac{1}{q_{k-1}}\right) \\ &= \frac{z_n + \mu n^{-1/2}}{(n+1)q_n} - \frac{1}{n+1}\sum_{k=1}^{n-1} (z_k + \mu k^{-1/2})\left(\frac{1}{q_{k+1}} - \frac{1}{q_k}\right) \\ &\quad - \frac{\tau_0}{n+1}\left(\frac{1}{q_1} - \frac{1}{q_0}\right). \end{aligned} \tag{15}$$

By (1) and (4),

$$\frac{1}{q_{k+1}} - \frac{1}{q_k} = \left(\frac{q_k}{q_{k+1}} - 1\right)\frac{1}{q_k} = O(1),$$

and consequently it follows from (14) that

$$n^{1/2}\sum_{k=1}^{n-1} k^{1/2}z_k k^{-1/2}\left(\frac{1}{q_{k-1}} - \frac{1}{q_k}\right) = o(1). \tag{16}$$

Next, by (14), (4), (15), and (16), we have that

$$s_n^1 + u_n^1 = o(n^{-1/2}),$$

where

$$u_n^1 := \frac{-\mu n^{-1/2}}{(n+1)q_n} + \frac{\mu}{n+1}\sum_{k=1}^{n-1} k^{-1/2}\left(\frac{1}{q_{k+1}} - \frac{1}{q_k}\right);$$

whence, for $n \geqslant 2$,

$$u_n = \frac{\mu}{q_n}((n-1)^{-1/2} - n^{-1/2}) = O(n^{-1/2}) = o(1),$$

by (4). It now follows, by Theorem C and the regularity of the members of

Γ, that $\{u_n\}$ and $\{s_n+u_n\}$ are summable to 0 by every member of Γ, and therefore so also is $\{s_n\}$. ∎

REFERENCES

1. D. BORWEIN, On methods of summability based on integral functions II, *Proc. Cambridge Philos. Soc.* **56** (1960), 125–131.
2. D. BORWEIN AND T. MARKOVICH, Cesàro and Borel-type summability, *Proc. Amer. Math. Soc.* **103** (1988), 1108–1111.
3. D. BORWEIN AND T. MARKOVICH, A tauberian theorem concerning Borel-type and Cesàro methods of summability, *Canadian J. Math.* **40** (1988), 228–247.
4. G. H. HARDY, "Divergent Series," Oxford Univ. Press, Oxford, 1949.
5. G. G. LORENTZ, Direct theorems on methods of summability, *Canad. J. Math.* **1** (1949), 305–319.
6. W. MEYER-KÖNIG, Untersuchungen über einige verwandte Limitierungsverfahren, *Math. Z.* **52** (1949), 257–304.
7. M. R. PARAMESWARAN, On summability functions for the circle family of methods, *Proc. Natl. Inst. Sci. India Part A* **25** (1959), 171–175.
8. M. R. PARAMESWARAN, Logarithmic means and summability by the circle methods, *Proc. Amer. Math. Soc.* **52** (1975), 279–281.
9. K. ZELLER AND W. BEEKMANN, "Theorie der Limitierungsverfahren," Zweite erweiterte und verbesserte Auflage, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 15, Springer-Verlag, Berlin/New York, 1970.